Laboratori Nazionali di Frascati

LNF-62/31 (1962)

N. Cabibbo, E. Ferrari: QUANTUM ELECTRODYNAMICS WITH DIRAC MONOPOLES.

Estratto dal: Nuovo Cimento, 23, 1147 (1962)

IL NUOVO CIMENTO Vol. XXIII, N. 6 16 Marzo 1962

Quantum Electrodynamics with Dirac Monopoles.

N. Cabibbo

Istituto di Fisica dell'Università - Roma
Laboratori Nazionali di Frascati del C.N.E.N. - Frascati

E. Ferrari

CERN - Geneva

(ricevuto il 16 Febbraio 1962)

1. – In a classical paper (1) Dirac has shown that quantum mechanics allows the existence of particles (monopoles) bearing a magnetic charge. The strength of the magnetic charge is not arbitrary: if monopoles must coexist with electrons, the allowed values are (2)

$$G = 2\pi n/e\,, \qquad (n \text{ integer}). \tag{1}$$

If different kinds of charged particles exist, eq. (1) must still be satisfied if we substitute their charge for the electron charge (possibly with different values of n). This means that the existence of monopoles would explain the empirical fact that the charges of elementary particles are all multiples of the electron charge e.

In this paper we discuss the extension of quantum electrodynamics to the case in which both fields with electric charge and monopole fields are present.

Previous theoretical treatments (3) made use of the usual representation of the e.m. field in terms of a vector potential A_μ:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\,. \tag{2}$$

The field produced by a magnetic point charge can be described in this way only if A_μ is allowed to be singular along an arbitrary line (string) starting from the pole and going to infinity.

This is clearly an unphysical feature, since the singularity in A_μ does not correspond to a singularity in the e.m. field $F_{\mu\nu}$.

---

(1) P. A. M. Dirac: *Proc. Roy. Soc.*, A **133**, 60 (1931); *Phys. Rev.*, **74**, 817 (1948).

(2) W use rationalized units with $\hbar = c = 1$, and a metric with $p^2 = |\boldsymbol{p}|^2 - p_0^2$.

(3) An extensive bibliography on the subject can be found in the paper by Bradner and Isbell and in the paper by Amaldi *et al.*, see footnote (5).

As we show in the following section, a non pathological description of the e.m. field produced by a given distribution of electric and magnetic sources can be obtained in terms of two vector potentials. The introduction of a second potential is compensated by an enlargement of the group of gauge transformations.

In the next two sections we build a quantized theory for the interactions of monopoles and charged particles, with the e.m. field without making use of potentials. This theory is an extension of the treatment recently given by S. MANDELSTAM (4) for the ordinary electrodynamics. Monopoles and charged particles are treated in a symmetrical way: the internal consistency of the theory requires the Dirac condition (eq. (1)).

In the last section we give a brief discussion of the symmetry properties of the theory. We show that, although parity is not conserved, parity non conservation effects can only appear if physical monopoles are present. The existence of monopoles (5) is therefore not contraddicted by the conservation of parity in ordinary electromagnetic processes, in which monopoles might take part as virtual particles.

2. – The Maxwell equations in vacuo can be written as (6):

$$\partial_\nu F_{\mu\nu}(x) = j_\mu(x)\,, \tag{3}$$

$$\partial_\nu \widetilde{F}_{\mu\nu}(x) = 0\,. \tag{4}$$

If sources of the magnetic field are allowed, eq. (4) should be substituted by:

$$\partial_\nu \widetilde{F}_{\mu\nu}(x) = g_\mu(x)\,, \tag{4'}$$

where the four-vector g_μ represents the magnetic current and the density of magnetic charge. Equations (3) and (4′) can be solved by means of two vector potentials, instead of one:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon^{\mu\nu\varrho\sigma}\partial_\varrho B_\sigma\,, \tag{5}$$

A_μ and B_μ are determined by $F_{\mu\nu}$ up to a group of gauge transformations; this contains individual gauge transformations, like

$$\begin{cases} A_\mu \to A_\mu + \partial_\mu \Lambda\,, \\ B_\mu \to B_\mu + \partial_\mu \Gamma\,, \end{cases} \tag{6}$$

as well as mixing transformations

$$\begin{cases} A_\mu \to A_\mu + A'_\mu\,, \\ B_\mu \to B_\mu + B'_\mu\,, \end{cases} \tag{7}$$

(4) S. MANDELSTAM: *Quantum Electrodynamics without Potentials*, preprint.

(5) Different experiments made up to now seem to exclude the existence of monopoles of mass $M < 2.5$ GeV: H. BRADNER and W. M. ISBELL: *Phys. Rev.*, **114**, 603 (1959); M. FIDECARO, G. FINOCCHIARO and G. GIACOMELLI: *Nuovo Cimento*, **22**, 657 (1961); E. AMALDI, G. BARONI, H. BRADNER, H. G. DE CARVALHO, L. HOFFMANN, A. MANFREDINI and G. VANDERHAEGHE: *1961 Conference on Elementary Particles at Aix-en-Provence*, vol. **1** (Salcay, 1962), p. 155. CERN Report, to be published.

(6) The symmetry between the electric field and the magnetic field is expressed by the duality operation $\widetilde{F}_{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\varrho\sigma}F_{\varrho\sigma}$ ($\varepsilon^{\mu\nu\varrho\sigma}$ the completely antisymmetric Ricci tensor, $\varepsilon^{1234} = i$). This operation is equivalent to the substitutions $\boldsymbol{E} \to \boldsymbol{H}$, $\boldsymbol{H} \to -\boldsymbol{E}$. Note that $\widetilde{\widetilde{F}}_{\mu\nu} = -F_{\mu\nu}$.

A' and B' shall satisfy the « zero field conditions »

$$\partial_\mu A'_\nu - \partial_\nu A'_\mu + \varepsilon^{\mu\nu\varrho\sigma}\partial_\varrho B'_\sigma = 0 \,. \tag{7'}$$

Note that transformations (6) are particular cases of (7).

We can use (6) to impose Lorentz conditions on A_μ and B_μ so that they will satisfy the following set of equations:

$$\begin{cases} \partial_\mu A_\mu = \partial_\mu B_\mu = 0 \,, \\ \Box A_\mu = j_\mu \,, \\ \Box B_\mu = g_\mu \,, \end{cases} \tag{8}$$

gauge transformations of the kind (6), with $\Box\Lambda = \Box\Gamma = 0$, or (7), with $\partial_\mu A'_\mu = \partial_\mu B'_\mu = 0$ are still allowed in the Lorentz gauge.

In the absence of sources ($g_\mu \equiv j_\mu \equiv 0$) we can adopt the usual gauge in which $B_\mu \equiv 0$, but we could equally well adopt a gauge in which $A_\mu \equiv 0$, or a general one as in eq. (5). The introduction of a second potential does not, due to the mixing transformations, cause an increase of the number of the independent variables which describe a free field. If we analise the free field in terms of photons we shall still have only two photons for each value of the linear momentum. The wave function of a given photon will however depend on the gauge adopted.

Any theory based on the general description (5) for the e.m. field, should be invariant under the whole of gauge transformations (6) and (7).

We note that if only monopoles and no charged particles were present one could adopt a description in terms of B_μ only. The resulting theory will be similar to ordinary electrodynamics, the only difference being in the higher value of the coupling constant (the minimum value allowed by eq. (1) is $g^2/4\pi = 34.25$). This treatment could be adequate for some problems like the annihilation of a monopole-anti-monopole pair into photons.

3. – MANDELSTAM has recently given a treatment of quantum electrodynamics in which no use is made of potentials (4). We extend this approach to the case in which both charged particles and monopoles are in interaction with the electromagnetic field. The Dirac condition (eq. (1)) for the electric and magnetic charges is necessary for the consistency of the theory; the theory is Lorentz-invariant and symmetrical between charges and monopoles.

To proceed by steps, we shall first consider the case of a charged scalar field, $\varphi(x)$, representing particles of electric charge e, in interaction with the electromagnetic field. Suppose, for the moment, that no magnetic sources exist, so that eqs. (3) and (4) hold, and the e.m. field can be described by a vector potential $A_\mu(x)$ (eq. (2)). Following MANDELSTAM, we introduce a new field quantity

$$\Phi(x; P) = \varphi(x) \exp\left[-ie\int_{(P)}^{x} A_\mu(\xi)\,\mathrm{d}\xi_\mu\right]. \tag{9}$$

The integral is evaluated on a spacelike path P ending at the point x. The new quantity $\Phi(x;P)$ does not depend on the gauge selected for A_μ but depends on the path P.

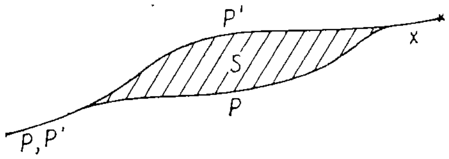

Fig. 1.

If we change the path from P to P' as in Fig. 1, Φ changes; from eq. (9):

$$\Phi(x, P') = \Phi(x, P) \exp\left[-ie \oint A_\mu \,\mathrm{d}\xi_\mu\right], \tag{9'}$$

the integral is now evaluated on the closed path $P'-P$; using the relativistic generalization of Stokes theorem:

$$\Phi(x, P') = \Phi(x, P) \exp\left[-\frac{ie}{2}\int_S F_{\mu\nu}\,\mathrm{d}\sigma_{\mu\nu}\right], \tag{10}$$

where S is a surface delimited by the path P'-P. Derivatives of $\Phi(x,P)$ correspond to the « gauge invariant derivatives » of $\varphi(x)$

$$\partial_\mu \Phi(x;P) = [(\partial_\mu - ieA_\mu)\varphi(x)] \exp\left[-ie\int_{(P)}^{x} A_\mu \,\mathrm{d}\xi_\mu\right]. \tag{11}$$

The derivatives of $\Phi(x,P)$ do not commute:

$$[\partial_\mu, \partial_\nu]\Phi(x,P) = \Phi(x,P)[-ie\,F_{\mu\nu}(x)]. \tag{12}$$

At this point we can forget eq. (9) and consider $\Phi(x,P)$ as defined by its path-dependence, which can be expressed directly in terms of the e.m. field $F_{\mu\nu}$ (eq. (10)) [7]. Equation (10) can be used in the general case in which magnetic sources are present and the e.m. field cannot be described (eq. (2)) in terms of a single potential. Its consistence requires however that the change in Φ does not depend on the particular choice of the 2-dimensional surface S. If S_1 and S_2 are two such surfaces:

$$\Phi(x,P)\exp\left[-\frac{ie}{2}\int_{S_1} F_{\mu\nu}\,\mathrm{d}\sigma_{\mu\nu}\right] = \Phi(x,P)\exp\left[-\frac{ie}{2}\int_{S_2} F_{\mu\nu}\,\mathrm{d}\sigma_{\mu\nu}\right],$$

([7]) Equation (12) can be derived directly from eq. (10), as shown in ([4]).

so that for the closed 2-dimensional surface $S = S_1 - S_2$ [8]:

$$\exp\left[-\frac{ie}{2}\int_S F_{\mu\nu}\,\mathrm{d}\sigma_{\mu\nu}\right] = 1\,.$$

We can change the surface integral to an integral over the volume V enclosed by S and, using eq. (4′), obtain:

$$\exp\left[-ie\int_V g_\mu\,\mathrm{d}V_\mu\right] = 1\,. \tag{13}$$

Equation (13) should hold for any volume V. Apart from the trivial case $g_\mu = 0$ considered by MANDELSTAM, other solutions exist, which correspond to the existence of Dirac monopoles:

(i) If g_μ is a classical (c number) source, eq. (13), requires that

$$Q_V = \int_V g_\mu\,\mathrm{d}V_\mu = \frac{2\pi n}{e}\,, \tag{14}$$

since V is completely arbitrary, eq. (14) can only be satisfied if g_μ is due to one or more pointlike sources, each with a magnetic charge multiple of $g = 2\pi/e$.

(ii) If g_μ is a quantum operator, eq. (13) is satisfied in operator form if all the eigenvalues of Q_V are multiple of g. This is true if g_μ represents the current of one or more quantized fields, each of them bearing a magnetic charge which is a multiple of g. These fields would then be associated with monopoles.

The Mandelstam scheme for the interaction of a charged field with the e.m. field can therefore be extended to the case in which monopoles exist, as long as their magnetic charges satisfy the Dirac condition (eq. (1)).

The extension of the scheme to the monopole fields is straightforward: a scalar monopole of magnetic charge g will be described by a path-dependent field quantity $\Psi(x, P)$. The path-dependence will be assumed to be given by [9]:

$$\Psi(x, P') = \Psi(x, P)\exp\left[-i\,\frac{g}{2}\int_S \tilde{F}_{\mu\nu}\,\mathrm{d}\sigma_{\mu\nu}\right]. \tag{15}$$

---

[8] The ordering of non commuting operators and the algebraic manipulations need some justification when the various quantities are quantized. In particular, it is sufficient to choose a particular ordering criterion and to follow it throughout the mathematical developments. In order to avoid the problems of commutativity between quantities calculated at points with a non-spacelike separation, one can restrict oneself to variations of the path P on a spacelike 3-dimensional surface Σ which contains x (*e.g.*, $t = \text{const}$) and to the choice of the 2-dimensional surface S lying on Σ. In this case we have only to deal with spacelike paths, surfaces, volumes. [It must be remarked, however, that eqs. (10) to (17) are not restricted to this case, but are valid in general. In this case all the commutation relations (20) to (25) for which the commutator is vanishing can be generalized to the form *e.g.* $[\Phi(x, P), \Phi(y, P')] = 0$ where it is understood that both x, y as well as the paths P, P' lie on a spacelike surface. For P, P' not satisfying this condition the path-dependence laws (10) and (15) have to be used explicitly.

[9] In a theory in which only monopoles are present we can describe the e.m. field $F_{\mu\nu}$ in terms of a potential B_μ (see Section **2**). In this case eq. (15) can be derived from a definition of $\psi(x, P)$ similar to eq. (9):

$$\Psi(x, P) = \psi(x)\exp\left[-ig\int_{(P)}^{x}\mathrm{d}\xi_\mu B_\mu(\xi)\right].$$

The derivatives will obey the commutation relations:

$$(16)\qquad [\partial_\mu, \partial_\nu]\,\Psi(x, P) = \Psi(x, P)\left|-i\frac{g}{2}\tilde{F}_{\mu\nu}(x)\right| .$$

In parallel to the case of a field with electric charge, the consistence of eq. (15) requires some constraint on the *electric* current j_μ:

$$(17)\qquad \exp\left[-ig\int_V j_\mu \,\mathrm{d}V_\mu\right] = 1 . \qquad \text{(any } V\text{)};$$

This condition can be satisfied if the electric charge is quantized.

4. – The scheme introduced in the last section for the description of charges and monopoles in interaction with the e.m. field does not contain pathological elements, like the string singularities. The path-dependence of the field variables is due to the fact that the space, in presence of an e.m. field, appears to a charged particle as curved [4].

In this section we will complete the scheme by postulating a set of equations of motion and commutation relations. We will proceed in three steps, considering cases in which: (i) only charged particles are present; (ii) only monopoles are present; (iii) both charges and monopoles are present. We note that the use of Lagrangians should be considered here only as an heuristic procedure. The problem of thed erivation of the equations given here from an action principle will be treated in a forthcoming paper.

For case (i) we follow the procedure given by MANDELSTAM [4]: from a Lagrangian (m is the mass of the charged particle)

$$(18)\qquad \mathscr{L} = -(\partial_\mu \Phi^*)(\partial_\mu \Phi) - m^2\Phi^*\Phi - \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} ,$$

we get the following equations of motion

$$(19)\qquad \Box\Phi - m^2\Phi = 0 , \qquad \partial_\nu F_{\mu\nu} = j_\mu = -ie[\Phi^*(\partial_\mu\Phi) - (\partial\Phi^*)\Phi] .$$

This set should be completed by eq. (4). As we have seen, eq. (4) cannot be considered as a necessary constraint (as Mandelstam does). We can nevertheless justify eq. (4) since we have shown that the only admissible inhomogeneous terms g_μ in eq. (4′) represent Dirac monopoles. From the Lagrangian (18) one can derive the following commutation relations [4]: (for equal times)

$$(20)\qquad \begin{cases} [\Phi(x, P),\ \Phi(y, P)] = [\Phi(x, P),\ \Phi^*(y, P)] = 0 , \\ [\dot\Phi(x, P),\ \dot\Phi(y, P)] = [\dot\Phi(x, P),\ \dot\Phi^*(y, P)] = 0 , \\ [\Phi(x, P),\ \dot\Phi(y, P)] = [\Phi^*(x, P),\ \dot\Phi^*(y, P)] = 0 . \end{cases}$$

$$(20') \quad \begin{cases} [\dot{\Phi}^*(x, P),\ \Phi(y, P)] = [\dot{\Phi}(x, P),\ \Phi^*(y, P)] = -i\,\delta^3(x-y)\,, \\ [\overset{(\cdot)}{\Phi}(x, P),\ F_{ij}(y)] = [\overset{(\cdot)}{\Phi}{}^*(x, P),\ F_{ij}(y)] = 0\,, \\ [\overset{(\cdot)}{\Phi}(x, P),\ F_{0i}(y)] = -e\displaystyle\int_{(P)}^{x} \mathrm{d}\xi_i\,\delta^3(y-\xi)\,\overset{(\cdot)}{\Phi}(x, P)\,, \\ [\overset{(\cdot)}{\Phi}{}^*(x, P),\ F_{0i}(y)] = e\displaystyle\int_{(P)}^{x} \mathrm{d}\xi_i\,\delta^3(y-\xi)\,\overset{(\cdot)}{\Phi}{}^*(x, P)\,. \end{cases}$$

A dot denotes differentiation in respect to t; when the dot is enclosed in brackets, as in eqs. (20′), the equation holds whether or not it is present.

The commutation relations for the components of F are the same as in the free field case:

$$(21) \quad \begin{cases} [F_{ij}(x),\ F_{i'j'}(y)] = [F_{0i}(x),\ F_{0i'}(y)] = 0\,, \\ [F_{0i}(x),\ F_{jk}(y)] = -\left\{\delta_{ij}\dfrac{\partial}{\partial y_k} - \delta_{ik}\dfrac{\partial}{\partial y_j}\right\}\delta^3(x-y)\,. \end{cases}$$

Case (ii) is related to case (i) by the duality operation [6], so that in this case the equations of motion will be (μ is the mass of the monopole)

$$(22) \qquad \Box\Psi - \mu^2\Psi = 0\,, \qquad \partial_\nu \widetilde{F}_{\mu\nu} = g_\mu = ig[\Psi^*(\partial_\mu \Psi) - (\partial_\mu \Psi^*)\Psi]\,,$$

plus eq. (3) with $j_\mu = 0$. In analogy with eqs. (20′) we will have:

$$(23) \quad \begin{cases} [\overset{(\cdot)}{\Psi}(x, P),\ \widetilde{F}_{ij}(y)] = [\overset{(\cdot)}{\Psi}{}^*(x, P),\ \widetilde{F}_{ij}(y)] = 0\,, \\ [\overset{(\cdot)}{\Psi}(x, P),\ \widetilde{F}_{0i}(y)] = -g\displaystyle\int_{(P)}^{x} \mathrm{d}\xi_i\,\delta^3(y-\xi)\,\overset{(\cdot)}{\Psi}(x, P)\,, \\ [\overset{(\cdot)}{\Psi}{}^*(x, P),\ \widetilde{F}_{0i}(y)] = g\displaystyle\int_{(P)}^{x} \mathrm{d}\xi_i\,\delta^3(y-\xi)\,\overset{(\cdot)}{\Psi}{}^*(x, P)\,. \end{cases}$$

The commutation relations among Ψ and Ψ^* can be obtained by eq. (20) substituting Ψ for Φ, Ψ^* for Φ^*.

The commutation relations among different components of $F_{\mu\nu}$ will still be given by eq. (21), since these are easily seen to be invariant under the duality operation.

We come now to the general case (iii) in which we have a field Φ bearing an electric charge e and a monopole field Ψ with magnetic charge g. In this case we see that a complete and coherent scheme is given by the path-dependence of the two fields, specified by eqs. (10) and (15), together with the equations of motion (19) and (22), and the commutation relations (20), (20′), (21) and (23).

It is easily seen that the consistence conditions expressed by eqs. (13) and (16) are automatically satisfied if (and only if) the charges e and g satisfy the Dirac condition (eq. (1)).

We will also assume that for equal times [10].

$$[\Psi(x, P),\ \Phi(y, P')] = 0\ . \tag{25}$$

5. – In the theory presented here the parity is not conserved; in fact if $F_{\mu\nu}$ is a tensor, $\partial_\nu \tilde{F}_{\mu\nu}$ is an axial vector, while g_μ is a vector. This is not surprising, since monopoles violate parity also in a classical theory: for instance in a magnetic field a monopole accelerates in the direction of the magnetic field. Also C, the conjugation of the electric charge, is not conserved. We find however new symmetries by combining the usual operations P and C with the reflection of the magnetic charge [11], M. Both $C' = CM$ and $P' = PM$ are conserved. In processes in which monopoles are not present as physical particles, P' and C' are equivalent to the usual operations P and C, and no parity (or C) violation is expected. The existence of monopoles is therefore not contraddicted by the observed parity conservation (and invariance under charge conjugation) in ordinary electromagnetic processes.

* * *

We are indebted to Prof. E. Amaldi who stimulated our interest in this subject, and to Professors R. Gatto, V. Glaser and L. Van Hove for discussions.

---

[10] This commutation relation is not changed by displacing any of the two paths on a space like surface [see footnote [8]] only if the Dirac condition (1) is satisfied.

[11] N. F. Ramsey: *Phys. Rev.*, **109**, 225 (1958).